1. Consider the linear dynamic system $\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x}$ with two state variables. For each matrix $\mathbf{A}$ below, create a partial phase portrait for each of the following systems by plotting trajectories beginning from (1,1), (1,0), (1,-1), (0,-1), (-1,-1), (-1,0), (-1,1), and (0,1). Determine the eigenvalues and eigenvectors of $\mathbf{A}$ and interpret them in light of your phase portrait (stable or unstable, type of fixed point).

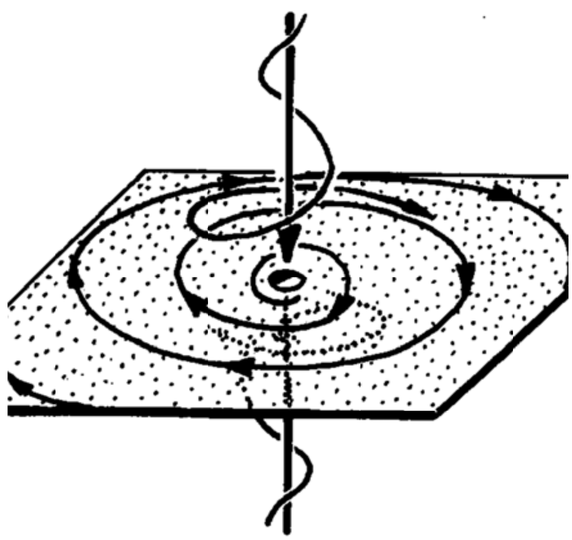
   a. $\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

   b. $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

   c. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

   d. $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}$

2. Using the following phase portrait for a linear system with three state variables, sketch the location of the eigenvalues in the complex plane. Briefly justify your choice of eigenvalue locations.



3. Consider this coupled system of ODEs:

$$\dot{x}_1 = 9x_1\left(1 - \frac{x_1}{9}\right) - 2x_1x_2$$
$$\dot{x}_2 = 6x_2\left(1 - \frac{x_2}{12}\right) - x_1x_2$$

This model captures the dynamics of two competing populations of bacteria. The two state variables represent the population densities of each species, the terms in parentheses cap the growth due to limitations in the environment, and the x_1x_2 terms represent the negative effects of competition between the species.

a. Show that the point $[\ 5\ 2\ ]^T$ is a fixed point.
b. Linearize the system around $[\ 5\ 2\ ]^T$ and find the eigenvalues and eigenvectors. Is this point stable or unstable? Is the local behavior oscillatory?
c. Sketch the phase portrait for this system, including the four fixed points, nullclines, and representative trajectories. Note that since the variables represent population densities, values less than zero are not meaningful and can be omitted from the diagram.
d. Briefly interpret the physical meaning of the phase portrait.

4. In the system below, identify the trivial fixed point and assess its linear stability. What type of fixed point is it? Then, use Matlab (or another numerical application) to plot the trajectory of the system in the x_2-x_3 plane for time up to at least 30 units starting from two close but different initial positions: (a) $[\ -11.5\ -12.3\ 30\ ]^T$ and (b) $[\ -11.5\ -12.3\ 30.01\ ]^T$. *Be careful with your numerics! I.e.*, be sure your step sizes are small enough to create a smooth trajectory. You may need to use a different integrator instead of `ode45`, or alter the settings of `ode45` to ensure a smooth result. (Bonus: use color to indicate the value of the third system variable x_1 at each point in the trajectory.) Compare the overall phase portraits obtained from (a) and (b), and compare the final (t = 30) state vectors. Can you make any observations about the nature of the system?
[Reference: Lorenz 1963 *J. Atmos. Sci.*]

$$\begin{aligned}\dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 &= -x_1x_3 + rx_1 - x_2 \qquad \sigma = 10,\ b = \tfrac{8}{3},\ r = 28 \\ \dot{x}_3 &= x_1x_2 - bx_3\end{aligned}$$